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A Zonal Boundary Element Method for Analysing Heat Exchangers with Thin Extended Surfaces

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Abstract—We present a zonal boundary element method (ZBEM) for the rapid, efficient, and accurate solution of the temperature distribution within extended surfaces whose aspect ratio may be asymptotically large. Our ZBEM employs a block-tridiagonal-matrix solution technique which decouples information which is 'far apart' with respect to the smallest dimension of the (asymptotically-thin) solution domain. We calculate asymptotic results for the performance advantage of the ZBEM over the classical BEM (CBEM), which are shown to be in excellent agreement with those of our numerical implementation. We propose that the ZBEM permits solution of asymptotically-large problems on modest computational platforms.

Keywords—Integral equations, Green's functions, Block-tridiagonal matrices.

1. INTRODUCTION

Most recuperative heat exchangers use a method of heat transfer wherein the interface separating two heat transfer fluids is a primary surface supplemented by a secondary surface in the form of slender metallic strips, called extended surfaces or fins, bonded to one side of the primary surface. Using such an arrangement, a large heat transfer surface can be achieved without using excessive amounts of primary surface, thereby minimising the size and/or weight of the heat exchanger.

The classical boundary element method (CBEM) has proved to be a 'natural' and useful tool [1–3] in the analysis of such heat exchangers, since it is able to represent accurately the complex inherent geometries. Despite this, the CBEM—and derivatives thereof (i.e., constant, linear, and quadratic elements)—has been restricted to those problems in which the length-to-width *aspect ratio*, hereafter denoted by $\mathcal{L}$, of the extended surfaces is artificially small, say $\mathcal{L} \sim O(1)$ to $O(10)$. However, the range $\mathcal{L} \sim O(100)$ to $O(1000)$ is more realistic.

It is well known that the CBEM relies on a dense-matrix solution procedure which links information at every boundary point with that at all other such points [4]. Thus, as $\mathcal{L}$ increases, the matrix becomes less and less diagonally dominant, resulting in an erosion of accuracy [5].

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Moreover, if the same mesh spacing is required in both the 'long' and the 'short' directions, there will clearly be both memory and CPU problems as $\mathcal{L}$ increases. A semi-analytic method has been suggested [6] in order to circumvent the problem of $\mathcal{L} \gg 1$; the method, which is a hybrid CBEM-asymptotic expansion approach, depends critically upon obtaining a far-field analytical solution which is then matched with the near-field CBEM solution.

In this paper, we present a zonal boundary element method (ZBEM) which is applicable to quite general geometries for which $\mathcal{L} \gg 1$. The ZBEM works on the principle of dividing the $\mathcal{L} \times 1$ solution domain into $\mathcal{L}$ coupled problems, each of which are solved on a 1×1 domain, to yield a block-tridiagonal-matrix solution technique. Thus, local diagonal dominance is preserved, and accuracy is nowhere eroded. We also demonstrate that, for the same boundary discretisation, the ZBEM beats the CBEM by a factor of $O(\mathcal{L}^2)$ in timing and $O(\mathcal{L})$ in storage; since $\mathcal{L} \gg 1$ in the problems we consider, these factors suggest a strong computational advantage of the ZBEM over the CBEM.

2. FORMULATION

The proposed ZBEM is most readily illustrated by way of a concrete example. We seek the temperature distribution $\tau(x, y)$ governed by the steady-state heat conduction equation

$$\Delta\tau(x, y) = 0 \tag{1}$$

in the (nondimensionalised) extended surface

$$(x, y) \in \Omega = [0, \mathcal{L}] \times [0, 1] \subset \mathbb{R}^2, \qquad \mathcal{L} \gg 1,$$

subject to the boundary conditions

$$\tau(0, y) = 5 - y^2, \qquad 0 \le y \le 1, \tag{2a}$$
$$\tau_y(x, 0) = 0, \qquad 0 \le x \le \mathcal{L}, \tag{2b}$$
$$\tau_x(\mathcal{L}, y) = 2\mathcal{L}, \qquad 0 \le y \le 1, \tag{2c}$$
$$\tau_y(x, 1) = -2, \qquad 0 \le x \le \mathcal{L}, \tag{2d}$$

so that (2a) can be considered to prescribe the primary-surface base temperature, whereas (2b)–(2d) represent convective heat fluxes across the unbonded faces of the extended surface. With subsequent analysis of errors in mind, both the extended surface geometry and conditions (2) have been chosen so that (1) and (2) admit the exact solution

$$\tau(x, u) = x^2 - y^2 + 5. \tag{3}$$

We stress that our ensuing ZBEM can be applied to far more complex geometries, but this would detract from the present discussion by complicating details. In the following discussion, we assume that the reader is familiar with the CBEM. If we apply the CBEM to (1) and (2), with N elements of length h in the y-direction and an element length in the x-direction of ρh, then the total number of elements is

$$N_C = 2N\left(1 + \frac{\mathcal{L}}{\rho}\right). \tag{4}$$

If $\rho = 1$, the resulting uniform mesh admits an error analysis, even if only at a heuristic level. Equation (4) now indicates that, if $\mathcal{L} \gg 1$, N_C will be prohibitively large unless ρ is accordingly raised (well) above unity, resulting in coarser resolution in the x-direction and, therefore, erosion of accuracy. From (4), the CBEM has leading-order storage requirement

$$S_C \sim O(N^2\mathcal{L}^2\rho^{-2}) \tag{5}$$

and, on the basis of matrix solution by LU decomposition, corresponding leading-order floating-point operation count

$$F_C \sim O(N^3 \mathcal{L}^3 \rho^{-3}). \tag{6}$$

Evidently, then, the CBEM can be made more efficient by increasing ρ, but this has the detrimental effect of reducing accuracy by coarsening the mesh in the x-direction; to circumvent this dilemma, we proceed as follows.

We subdivide the original solution domain $\Omega = [0, \mathcal{L}] \times [0, 1]$ into $\mathcal{L}$ zones $\Omega_\lambda = [\lambda - 1, \lambda] \times [0, 1]$ $(\lambda = 1, \ldots, \mathcal{L})$, each of which has unit aspect ratio, so that

$$\Omega = \bigcup_{\lambda=1}^{\mathcal{L}} \Omega_\lambda. \tag{7}$$

The ZBEM applies the CBEM to each zone Ω_λ (with N elements of length h on each side), after having introduced 'virtual' internal boundaries, between $\Omega_{\lambda-1}$ and Ω_λ, on which information is not specified *a priori* via the physical boundary conditions. From *a posteriori* reasoning, we remark that elements in each zone should be numbered counterclockwise in the following sequence: west face, south face, north face, east face. If, then, the N-vectors $\boldsymbol{\tau}^E_{\lambda-1}$ and $\delta\boldsymbol{\tau}^E_{\lambda-1}$ represent temperatures and fluxes at *all elements* on the (virtual) east face of $\Omega_{\lambda-1}$, whereas $\boldsymbol{\tau}^W_\lambda$ and $\delta\boldsymbol{\tau}^W_\lambda$ represent corresponding information on the west face of Ω_λ, then clearly, for all $\lambda = 2, \ldots, \mathcal{L} - 1$,

$$\boldsymbol{\tau}^W_\lambda = \mathbf{J}_N \, \boldsymbol{\tau}^E_{\lambda-1} \tag{8}$$

and

$$\delta\boldsymbol{\tau}^W_\lambda = -\mathbf{J}_N \, \delta\boldsymbol{\tau}^E_{\lambda-1}, \tag{9}$$

where $\mathbf{J}_N$ is the $N \times N$ 'reversal' matrix, with ones on the trailing diagonal and zeroes elsewhere.

Thus, the ZBEM transforms our original problem into one of inverting $\mathcal{L}$ matrices, of size $4N \times 4N$, coupled via equations (8) and (9); fortunately, this coupling is localised, so that we do not have to solve an overall $4N\mathcal{L} \times 4N\mathcal{L}$ system (as per the CBEM). A more lengthy analysis [7] reveals that our approach enables us to write the ZBEM system in the *block-tridiagonal* form

$$\mathbf{M}_{1\lambda} \boldsymbol{\beta}_{\lambda-1} + \mathbf{M}_{2\lambda} \boldsymbol{\beta}_\lambda + \mathbf{M}_{3\lambda} \boldsymbol{\beta}_{\lambda+1} = \boldsymbol{\alpha}_\lambda, \qquad \lambda = 1, \ldots, \mathcal{L}, \tag{10}$$

where $\boldsymbol{\beta}_\lambda$ represents the $4N$-vector of unknowns (ordered ESNW) on the boundary of Ω_λ, $\boldsymbol{\beta}_0 = \boldsymbol{\beta}_{\mathcal{L}+1} = \mathbf{0}$, and $\boldsymbol{\alpha}_\lambda$ is the $4N$-vector arising through the enforcement of the physical conditions on the external boundaries. For the test problem described by (1) and (2), the $4N \times 4N$ matrices $\mathbf{M}_{1\lambda}$, $\mathbf{M}_{2\lambda}$, and $\mathbf{M}_{3\lambda}$ are actually independent of λ (because of the geometric self-similarity of the zones), thereby reducing computational cost in evaluation of the coefficients, which are the 'usual' element-by-element integrations of products of trial functions and logarithmic Green's functions.

The block-tridiagonal system (10) can be solved via block LU factorisation [8] provided that the $\mathcal{L}$ submatrices

$$[\mathbf{M}_{21}], \quad \begin{bmatrix} \mathbf{M}_{21} & \mathbf{M}_{31} \\ \mathbf{M}_{12} & \mathbf{M}_{22} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{M}_{21} & \mathbf{M}_{31} & \mathbf{0}_N \\ \mathbf{M}_{12} & \mathbf{M}_{22} & \mathbf{M}_{32} \\ \mathbf{0}_N & \mathbf{M}_{13} & \mathbf{M}_{23} \end{bmatrix}, \quad \text{etc.}$$

(where $\mathbf{0}_N$ is the $4N \times 4N$ zero matrix) are nonsingular. Given both the generality of domain geometry and dependence of matrix coefficients thereon, it is far from clear whether necessary and sufficient conditions for nonsingularity can be proved explicitly; in the absence of such proof, we report that we encountered no problems when implementing the above algorithm.

The block LU factorisation thus comprises $\mathcal{L}$ solutions of a $4N \times 4N$ system, with corresponding leading-order 'flop' estimate

$$F_Z \sim O(N^3\mathcal{L}), \tag{11}$$

which on comparison with (6) reveals that the ZBEM is, in theory, $O(\mathcal{L}^2)$ times faster than the CBEM. The above description also reveals that the storage requirement for the ZBEM is

$$S_Z \sim O(N^2\mathcal{L}) \tag{12}$$

so that the CBEM requires $O(\mathcal{L})$ times the storage of the ZBEM. More accurate estimates of F_C, S_C, F_Z, and S_Z are presented in [7].

3. RESULTS AND DISCUSSION

Both the CBEM and ZBEM were used to solve the test problem described by (1) and (2). In the notation of the above section, results were obtained by varying the parameters $\mathcal{L}$ and N. All calculations were performed on a Sun SPARCsystem 400 with 32 MB RAM, on which one addition/multiplication flop required 1.0 μs of CPU time. With $\rho = 1$ (i.e., both methods employing a uniform mesh) and $N = 4$, the CBEM storage requirements exceeded the 32 MB RAM when $\mathcal{L} \sim 250$, whereas the ZBEM could still solve up to $\mathcal{L} \sim 12500$.

Numerical solutions (in an obvious notation) τ_C and τ_Z were obtained at M points (x_i, y_i) on a regular mesh throughout the overall domain Ω. Thereafter, the mean relative errors

$$\varepsilon_C = \frac{1}{M}\sum_{i=1}^{M}\left|1 - \frac{\tau_C(x_i, y_i)}{\tau(x_i, y_i)}\right| \tag{13}$$

and

$$\varepsilon_Z = \frac{1}{M}\sum_{i=1}^{M}\left|1 - \frac{\tau_Z(x_i, y_i)}{\tau(x_i, y_i)}\right|, \tag{14}$$

where τ is given in (3), were calculated for $\rho = 1$, $N = 4$ and various values of $\mathcal{L} \gg 1$; results are presented in Table 1. The blank field in Table 1 arises since the $(\mathcal{L}, \rho, N) = (1000, 1, 4)$ parameter triad is beyond the capability of the 32 MB platform on which the calculations were performed. Table 1 reveals that the ZBEM mean errors are consistently smaller than the corresponding CBEM figures; this is due to the fact that, for $(x_i, y_i) \in \Omega_\lambda$, the evaluation of $\tau_Z(x_i, y_i)$ is performed using the information on only the boundary of the *local* Ω_λ, and so minor errors in the resolved boundary values are not amplified via multiplication by large coefficients from 'distant' elements. One might guess that the benefits of localisation might be countered by the ZBEM's introduction of several interior boundaries, on which we expect numerical errors to be greatest via the Maximum Principle [9]; our numerical evidence would suggest that the advantages of zoning outweigh the errors *potentially* introduced via interior discretisation.

Table 1. Mean relative errors, with increasing $\mathcal{L}$, for $\rho = 1$ and $N = 4$.

$\mathcal{L}$	ε_C	ε_Z
10	2.88×10^{-4}	2.56×10^{-4}
100	2.06×10^{-5}	1.41×10^{-5}
1000	—	1.31×10^{-6}

Perhaps the most important prediction regarding the ZBEM is that of reducing drastically solution times for $\mathcal{L} \gg 1$. In Table 2, we present both the CPU times obtained on the above-described platform and those based upon the (lengthy) theoretical analysis in [7], which includes not only arithmetical factors, but also those due to array assignments and subscripting overheads.

The final column in Table 2 demonstrates clearly that, for $\mathcal{L} = 100$, the ZBEM is approximately 250 times faster than the CBEM, a practical result substantiated rather accurately by the theory in [7]. When $\mathcal{L} = 1000$, the 'advantage' ratio for the ZBEM would be a remarkable 19600, were it not for the fact that the CBEM cannot be used in view of storage restraints. It is also interesting to note not only the ratio of times from both methods, but also the actual times themselves; e.g, when $\mathcal{L} = 100$ (a perfectly realistic value), the CBEM requires almost 10 minutes of CPU compared to the ZBEM's 2 seconds.

Table 2. CPU times (in seconds) required for inversion of system matrix, and 'advantage' ratios for $\rho = 1$ and $N = 4$. Theoretical estimates [7] (in parentheses) for comparison.

$\mathcal{L}$	T_C	T_Z	$\frac{T_C}{T_Z}$
10	0.74 (0.72)	0.20 (0.25)	3.70 (2.92)
100	556 (540)	2.11 (2.65)	265 (203)
1000	— (524000)	22.2 (26.7)	— (19600)

4. CONCLUSIONS

We have presented a novel zonal boundary element method (ZBEM) for the solution of heat transfer problems within extended surfaces whose aspect ratio may be asymptotically large. We have demonstrated that the ZBEM is efficient, accurate, and extremely rapid, and that it also permits solution, on modest computational platforms, of problems which were hitherto beyond the reach of all but the largest computers. The authors acknowledge that the ZBEM could be made yet more rapid via the use of specialised matrix techniques for block-tridiagonal systems and/or a parallel implementation, but that such concepts are beyond the scope of the present study.

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